A false oscillation with a wave length of two times the grid size arises from a finite difference advection-diffusion equation if the grid size is greater than a critical value determined by the coefficient of subgrid-scale diffusion.

For simplicity, a one-dimensional, steady state advection-diffusion equation is dealt with:

\[ u \frac{\partial^2 T}{\partial x^2} = -B \frac{\partial^3 T}{\partial x^4}, \quad (1) \]

where \( u \) is the \( x \)-component of the velocity assumed to be positive, \( T \) is a state variable, say, temperature, and \( B \) is the coefficient of subgrid-scale diffusion. Both \( u \) and \( B \) are assumed to be constant.

Instead of the biharmonic form \(-B \partial^4 T / \partial x^4\), the diffusion term is conventionally written in harmonic form as \( A \partial^2 T / \partial x^2 \).

The relationship between \( A \) and \( B \) is determined as follows (SEMTNER and MINTZ, 1977).

When centered differencing is used, both terms are approximated by

\[ -B \frac{\partial^3 T}{\partial x^4} = -B \times \frac{\sum_{i=-n}^{n} T_{n+i} + 6T_n - 4T_{n-1} + T_{n-2}}{(Ax)^4}, \quad (2) \]

\[ A \frac{\partial^2 T}{\partial x^2} = A \times \frac{T_{n+1} - 2T_n + T_{n-1}}{(Ax)^2}, \quad (3) \]

where \( Ax \) is the grid size, \( T_{n+2}, T_{n+1} \) and \( T_n \) are the temperatures at the \( (n+2), (n+1) \) and \( n \)th grid points. With \( T = q e^{ikx} (i = \sqrt{-1}), \) the right-hand side of (2) becomes

\[ -q \frac{16B}{(Ax)^4} \left( \sin \frac{kAx}{2} \right)^4 \quad \text{and that of (3) becomes} \]

\[ -q \frac{4A}{(Ax)^3} \left( \sin \frac{kAx}{2} \right)^3. \quad (5) \]

The shortest wave length resolvable by a grid size of \( Ax \) is \( 2Ax \), so that \( \sin (kAx/2) = 1 \). If the magnitude of (4) is made equal to that of (5) at this shortest wave length, then it follows that

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Relationship between the grid size and the coefficient

\[ B = (\Delta x)^2 A / 4 \quad (6) \]

When the amplification factor is denoted by \( \delta \), the centered differencing of Eq. (1) leads to

\[ \delta^4 + (0.5d - 4) \delta^3 + 6 \delta^2 - (0.5d + 4) \delta + 1 = 0, \quad (7) \]

where

\[ d = (\mu / \rho)^{1/3} \Delta x. \quad (8) \]

Figure 1 shows the four roots \( \delta_m \) \((m = 1, \ldots, 4)\) as functions of \( d \). The real and imaginary parts are shown for \( \delta_1 \) and \( \delta_5 \) if these are complex numbers.

On the other hand, substitution of \( T \propto e^{u^2} \) into (1) leads to

\[ u \sigma = -B \sigma^4, \]

which gives

\[ \sigma_1 = 0, \]
\[ \sigma_2 = -\left( (\mu / \rho)^{1/3} \right)^{1/2} \]
\[ \sigma_{3,4} = \left( \mu / \rho \right)^{1/2} \frac{1 \pm \sqrt{3} i}{2}. \]

The amplification factors \( \delta_\sigma \) to be compared with \( \delta_m \) are given by \( \exp (\Delta x \sigma_m) \) \((m = 1, \ldots, 4)\), which are also shown in Fig. 1. The real and imaginary parts are shown for \( \delta_1 \) and \( \delta_5 \).

Obviously \( \delta_1 \) is identical with \( \delta_1' \) (\( \delta_5 = \delta_5' = 1 \)) irrespective of \( d \). No significant difference is found between \( \delta_1 \) and \( \delta_5 \) in the practical range of \( d \). As is readily seen however, \( \delta_1 \) and \( \delta_5 \) are qualitatively different from \( \delta_1' \) and \( \delta_5' \) for \( d < 2.748 \), although they agree well with \( \delta_1' \) and \( \delta_5' \) for \( d < 2.0 \). Both \( \delta_1 \) and \( \delta_5 \) are real, negative numbers for \( d > 2.748 \), which gives rise to a false oscillation with a wave length of \( 2 \Delta x \).

The condition necessary for getting a solution of Eq. (1) turns out to be

\[ d < 2.748, \quad (9) \]

or with (6) and (8),

\[ 0.193 \mu \Delta x < A \quad \text{(10)} \]

When the diffusion term is formulated by \( A \delta^2 T / \partial x^2 \), the condition (TAKANO, 1974) corresponding to (10) is

\[ 0.5 \mu \Delta x < A. \quad (11) \]

Therefore, compared with harmonic diffusion, biharmonic diffusion allows a smaller \( A \) for a given \( \Delta x \), or a larger \( \Delta x \) for a given \( A \). This is an advantage of \( -B \delta^2 T / \partial x^2 \) over \( A \delta^2 T / \partial x^2 \). As already pointed out (SEMNTNER and MINTZ, 1977), the primary advantage is that the biharmonic formulation is highly scale selective because of a factor of \( (\sin k \Delta x / 2)^4 \) in (4) in place of a factor of \( (\sin k \Delta x / 2)^2 \) in (5); the ratio of \( -B \delta^2 T / \partial x^2 \) to \( A \delta^2 T / \partial x^2 \) is 0.065 for a wave length of 10\( \Delta x \) and 0.024 for a wave length of 20\( \Delta x \). Biharmonic formulation brings about very weak diffusion for wave lengths larger than 2\( \Delta x \).

References
