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Island-trapped shelf waves*

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Abstract : A theoretical model is presented of low-frequency oscillations around an island with sloping beach. It is shown that shelf waves with various radial modes and azimuthal wavenumbers can be trapped, but there exists a low wavenumber cut-off for each trapped mode. The results are compared with observations from the Hawaiian Islands.

I. Introduction

The trapping of low-frequency (i.e., less than inertial frequency) waves by islands has been a topic of much interest in the past decade, although it has been studied less extensively than trapping by straight coastlines. A good summary of the topic is contained in a review paper by MYSAK (1979). MIYATA and GROVES (1968) detected a two-day oscillation around the island of Oahu by analyzing the tide gauge records from Honolulu and Mokuoloe. CALDWELL and EIDE (1976) verified in a laboratory experiment that such an oscillation could in fact be excited around Oahu as a shelf wave.

The first theoretical study of island-trapped, sub-inertial waves was made by MYSAK (1967), but his theory applied only for a large island (modelling Australia) with a narrow, sloping shelf in the non-divergent limit (rigid lid approximation). On the other hand, RHINES (1969) suggested that near a small island with sloping sides it was possible for trapped oscillations of shelf wave type to exist. A more complete study of island-trapped shelf waves was made by LONGUET-HIGGINS (1970). He considered an island of circular symmetry when the depth (h) was given as a function of the radial distance (r),

$$h=\begin{cases} h_1(\frac{r}{a})^{\alpha}, & a\leq r<b \\ h_1(\frac{b}{a})^{\alpha}, & b\leq r \end{cases} \qquad (1.1)$$

where (a) is the radius of the island, (b−a) the shelf width, and (α) any constant. On the other hand, SAINT-GUILY (1972) examined the trapped low-frequency modes for an island with para-bolically sloped beach that extends to infinity ∞:

$$h = h_1(r^2-a^2),\ a\leq r\leq\infty \qquad (1.2)$$

This model also produces trapped shelf wave solutions. The results of SAINT-GUILY, however, differ quite distinctly from those of Longuet-Higgins. That is, the model topography (1.2) produces only a finite number of trapped modes for a given azimuthal wavenumber, whereas the model (1.1) yields an infinite number of modes for each wavenumber. The reason for this discrepancy has not been discussed as yet.

Three major differnces between the models of LONGUET-HIGGINS (1.1) and SAINT-GUILY (1.2) are:

1. Model (1.1) has a surrounding ocean of finite depth whereas (1.2) has an infinitely deep ocean;
2. A beach with shoreline (h = 0 at r = a) is included in (1.2) but not in (1.1) which has a vertical wall at the perimeter of the island;
3. LONGUET-HIGGINS used the rigid lid approximation, whereas Saint-Guily allowed horizontal divergence.

The main purpose of this paper is to examine the trapped modes for a model island that is a combination of (1.1) and (1.2), and to determine which of the three differences is the source of the aforementioned discrepancy. The model

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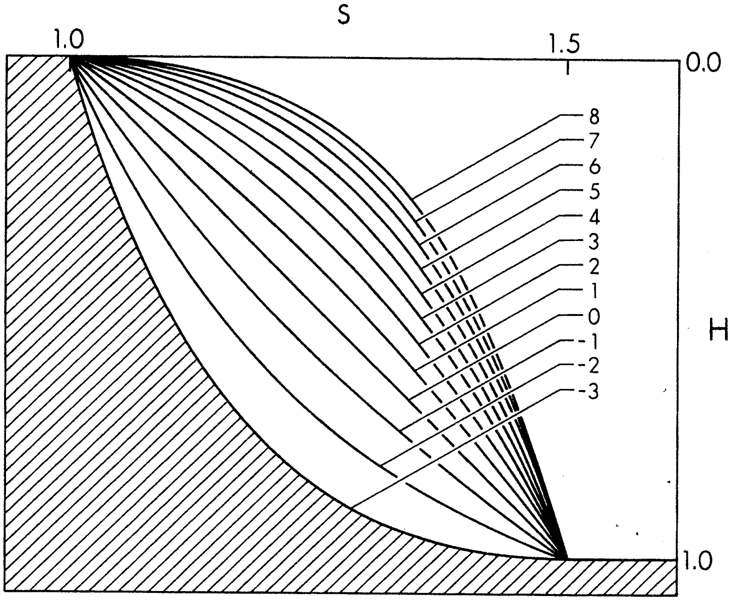

Fig. 1. Examples of cross-section of the model island shelf given by (2.11) for B=1.5. Numbers (8,7, ...,-3) indicate values of δ.

employed here is:

$$h=\begin{cases} h_1 r^{\delta}(r-a) & a\leq r\leq b, \\ h_1 b^{\delta}(b-a) & b\leq r \end{cases} \quad (1.3)$$

where δ is an arbitrary constant.

Modelling and formulation are explained in the following section. The solutions are given in Section III. It will be shown that the presence of the shoreline in the model (1.3) plays an essential role. The results obtained will be discussed and applied to observations from the islands of Oahu and Hawaii.

2. Formulation

The linear unforced shallow water equations for a homogeneous, inviscid ocean with an axisymmetric depth profile h(r) are given in cylindrical coordinates:

$$\frac{\partial u}{\partial t}-fv = -g\frac{\partial \zeta}{\partial r}, \quad (2.1)$$

$$\frac{\partial v}{\partial t}+fu = -g\frac{1}{r}\frac{\partial \zeta}{\partial \theta}, \quad (2.2)$$

$$\frac{\partial \zeta}{\partial t}+\frac{1}{r}\frac{\partial (rhu)}{\partial r}+\frac{1}{r}\frac{\partial (hv)}{\partial \theta} = 0, \quad (2.3)$$

where (u) and (v) are radial and azimuthal velocities, (ζ) is surface displacement, (g) is gravity acceleration, and (f) denotes the constant Coriolis parameter. Introducing the non-dimensional variables, $t=f^{-1}t'$, $r=as$, $h=h_0H(s)$, where (a) is the radius of the island and (h_0) is the depth as $r\rightarrow\infty$, and assuming the wave solutions around the island, $(u,v,\zeta)=(-i\sqrt{gh_0}U, \sqrt{gh_0}V, h_0Z)e^{i(n\theta-\sigma/ft')}$, then the equations (2.1), (2.2) and (2.3) become

$$\omega U + V = \frac{1}{\mu}\frac{dZ}{ds}, \quad (2.4)$$

$$\omega V+U = \frac{n}{\mu s}Z, \quad (2.5)$$

$$\omega Z+\frac{1}{\mu s}\frac{d}{ds}(sHU)-\frac{n}{\mu s}HV = 0, \quad (2.6)$$

where $\omega=\frac{\sigma}{f}$ is the non-dimensional frequency, and the variables U, V, Z are functions of (s) only. The parameter (μ) is the ratio of the radius of the island to the Rossby radius of deformation,

$$\mu = \frac{af}{\sqrt{gh_0}} \quad (2.7)$$

From (2.4) and (2.5) we obtain

$$U=\frac{1}{\mu(1-\omega^2)}\left(\frac{n}{s}Z-\omega\frac{dZ}{ds}\right), \quad (2.8)$$

$$V=\frac{1}{\mu(1-\omega^2)}\left(\frac{dZ}{ds}-\frac{n\omega}{s}\cdot Z\right). \quad (2.9)$$

Substituting (2.8) and (2.9) into (2.6) yields

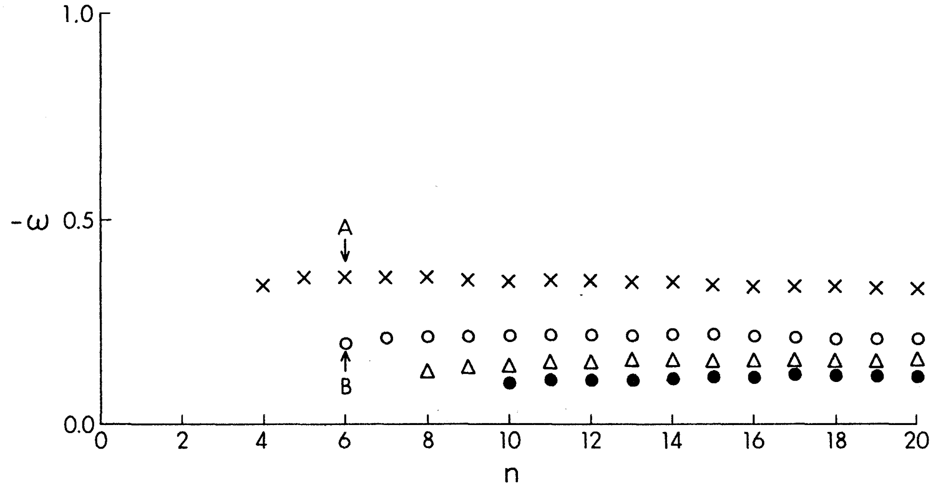

Fig. 2. Non-demensional despersion relation for $\delta=0$, B=1.8, $\mu=0.01$. P and Q are for Fig. 3. 〔× : 1st mode; ○ : 2nd mode; △ : 3rd mode; ● : 4th mode〕

$$\frac{d}{ds}(sH\frac{dZ}{ds})-\left\{\mu^2(1-\omega^2)s+\frac{n^2H}{s}\right.$$
$$\left.+\frac{n}{\omega}\frac{dH}{ds}\right\}Z=0. \qquad (2.10)$$

The depth profile (1.3) is now in the non-dimensional form:

$$H=\begin{cases}Cs^{\delta}(s-1) & 1\leq s\leq B \\ 1 & B\leq s\end{cases} \qquad (2.11)$$

where $C=B^{-\delta}(B-1)^{-1}$ and $B=\frac{a}{b}$. Some example profiles are given in Fig. 1. Now the problem is to solve Eq. (2.10) with the boundary conditions:

$$Z, U, V \rightarrow O, \text{ as } s\rightarrow\infty, \qquad (2.12)$$
$$UH=O, \text{ at } s=1. \qquad (2.13)$$

The condition (2.12) is required because we are seeking trapped waves only, and (2.13) specifies no normal flow condition at the island.

3. Solutions and Discussions

For $s\geq B$, Eq. (2.10) is simply

$$\frac{d}{ds}(s\frac{dZ}{ds})-\{\mu^2(1-\omega^2)s/H+\frac{n^2}{s}\}Z=0. \qquad (3.1)$$

This is a Sturm-Liouville type equation and the character of the solution is determined by the sign of the second term. As $\mu^2(1-\omega^2)s/H$ dominates over n^2/s sufficiently far from an island or shelf, $(1-\omega^2)$ must be positive in order for the solutions to be trapped. The solution can be expressed as

$$Z_I=A_IK_n(n\sqrt{\frac{1-\omega^2s}{H}}) \qquad (3.2)$$

where K_n is the modified Bessel function of the order (n) and (I) indicates the region outside the shelf.

It is assumed that for any s, $1\leq s\leq B$,

$$\mu^2(1-\omega^2)\ll\frac{n^2H}{s^2} \text{ or } \left|\frac{n}{\omega s}\frac{dH}{ds}\right|. \qquad (3.3)$$

This assumption is equivalent to neglecting horizontal divergence in this region (rigid lid approximation). Now equation (2.10) can be rewritten as

$$\frac{d}{ds}s^{\delta+1}(s-1)\frac{dZ}{ds}-\left\{n^2s^{\delta-1}(s-1)\right.$$
$$\left.+\frac{n}{\omega}\frac{d}{ds}s^{\delta}(s-1)\right\}Z=0. \qquad (3.4)$$

Then the appropriate solution over the shelf region is found(see e.g. GOLDSTEIN and BROWN, 1973) to be

$$Z_{II}=A_{II}s^{-\ell-\frac{\delta}{2}}F(\ell+m, \ell-m, 1; 1-\frac{1}{s}), \qquad (3.5)$$

where (II) indicates the region over the shelf and (F) denotes the hypergeometric function with:

$$\ell=\frac{1}{2}(1-\sqrt{4n^2+\frac{4n}{\omega}(1+\delta)+(1+\delta)^2})$$

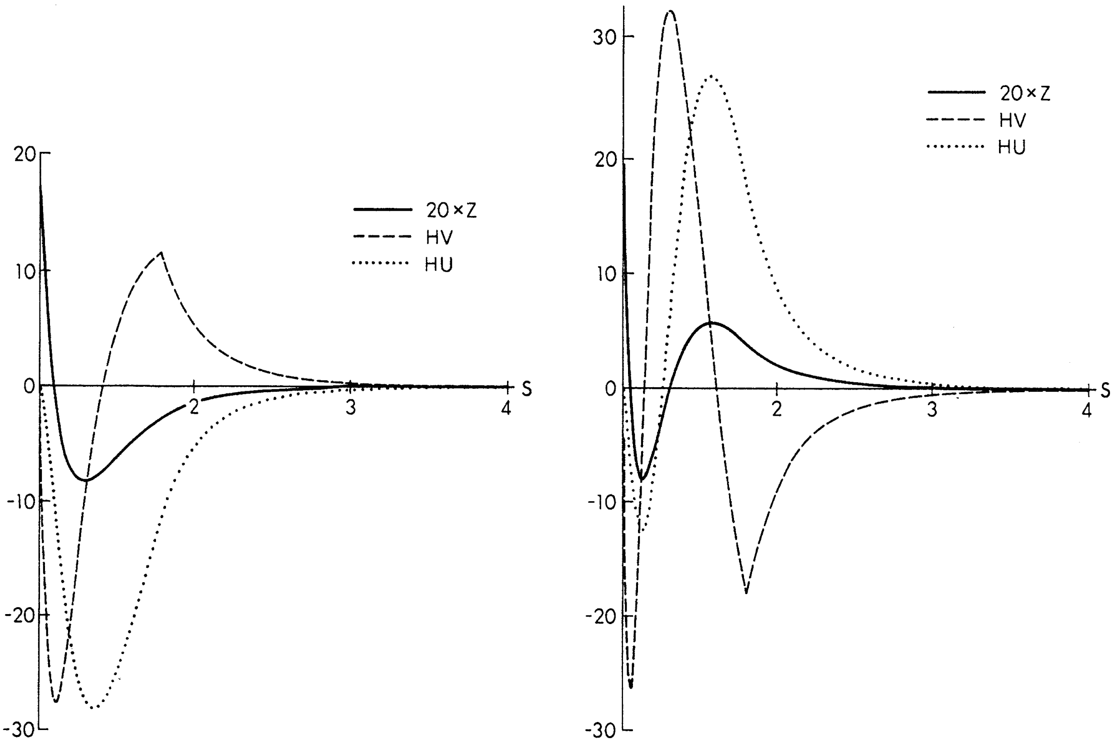

Fig. 3. (a) Non-dimensional sea level Z, radial and azimuthal volume transports HV and HU for the wave P in Fig. 2.
(b) Non-dimensional sea level Z, radial and azimuthal volume transports HV and HU for the wave Q in Fig. 2.

$$m=\frac{1}{2}\sqrt{4n^2+\frac{4n\delta}{\omega}+\delta^2}.$$

At the edge of the shelf s=B, both the displacement and velocities must be continuous, so that

$$Z_I=Z_{II} \quad \text{at} \quad s=B, \tag{3.6}$$

$$\frac{dZ_I}{ds}=\frac{dZ_{II}}{ds} \quad \text{at} \quad s=B. \tag{3.7}$$

These conditions yield

$$\frac{dK_n}{ds}F=K_n\left\{\frac{dF}{ds}-(1-\frac{\delta}{2})s^{-1}F\right\},$$

$$\text{at} \quad s=B. \tag{3.8}$$

The equation (3.8) determines the eigenvalue ω for a given n, thus providing the dispersion relation. Fig. 2 shows a dispersion relation when $\delta=0.$, B=1.8, and $\mu=0.01$. It is seen that ω is always negative (corresponding to right-bounded waves) and the absolute value of ω is less than unity. In this sense, the result is similar to that of continental shelf waves for a straight coastline (MYSAK, 1968) with a sloping beach. Island-trapped shelf waves, however, have two unique properties. One is that as there is an integral number of wavelengths around the island, the dispersion relationship exists only for discrete wavenumbers. The other unique property is that there is a low wavenumber cut-off in the spectrum. In Fg. 2, the lowest possible azimuthal wavenumber for the first radial mode is 4. It can be seen that the higher the radial mode number, the greater the cut-off wavenumber. It is interesting to note that MYSAK'S (1968) solutions show similar low wavenumber cut-off in the trapped gravity wave spectrum but not in the shelf wave region. This property does not appear in LONGUET-HIGGINS' result (1970), which shows that for given n, there exist an infinite number of

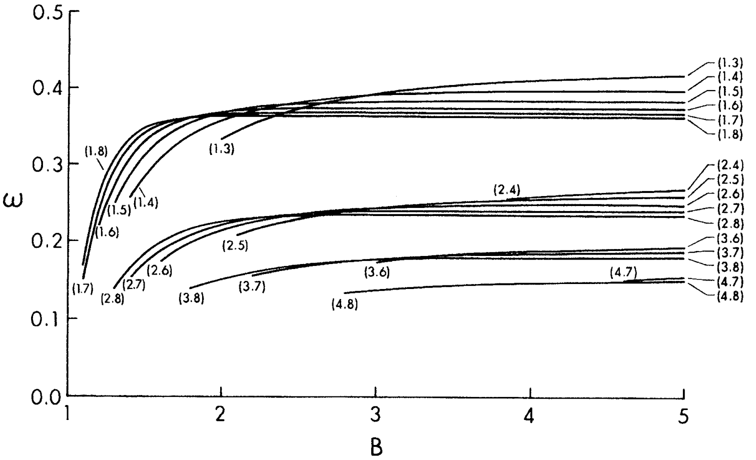

Fig. 4 (a). Non-dimensional frequency ω as a function of the non-dimensional shelf width B for $\delta=0$ and $P=0.01$. The symbol (j, n) represents the j-th mode with azimuthal wavenumber n.

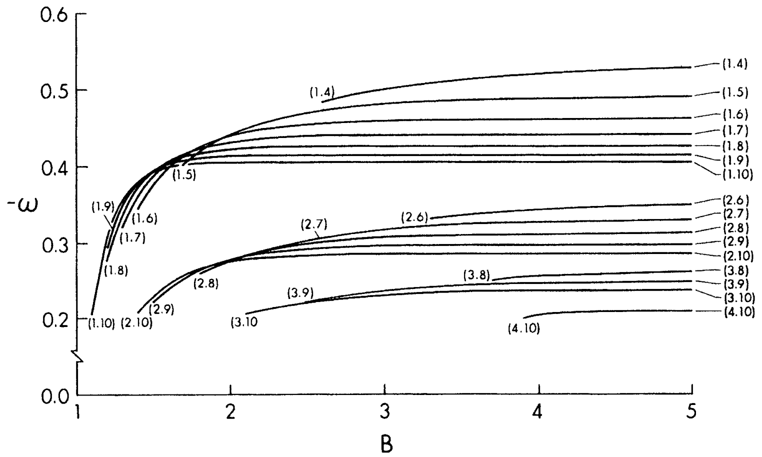

Fig. 4 (b). Same as Fig. 4(a), but for $\delta=1$.

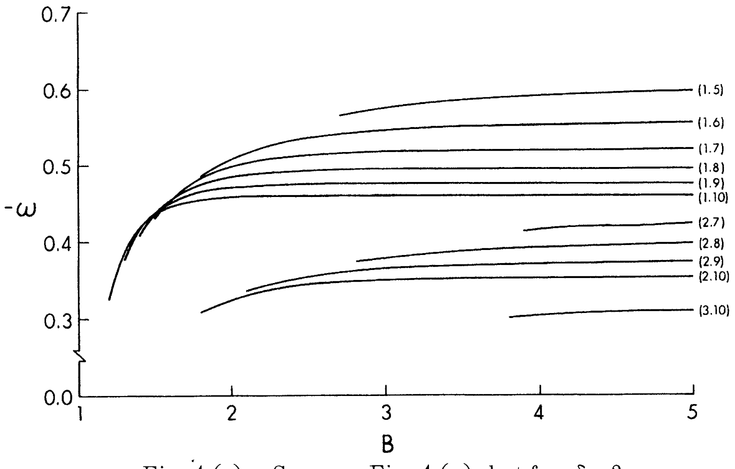

Fig. 4 (c). Same as Fig. 4 (a), but for $\delta=2$.

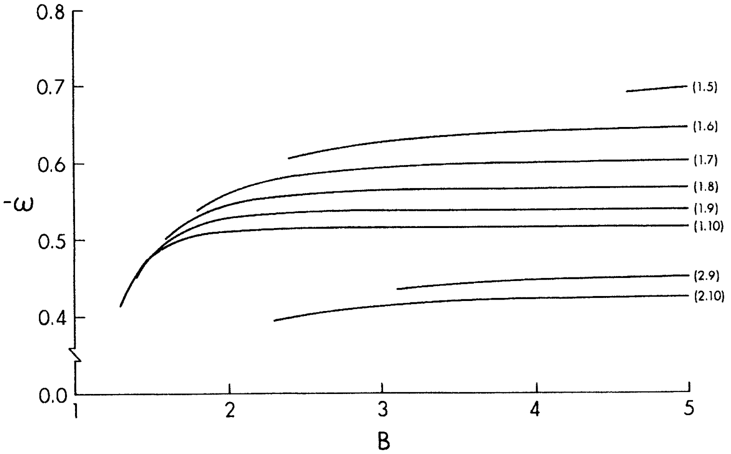

Fig. 4 (d). Same as Fig. 4 (a), but for $\delta=3$.

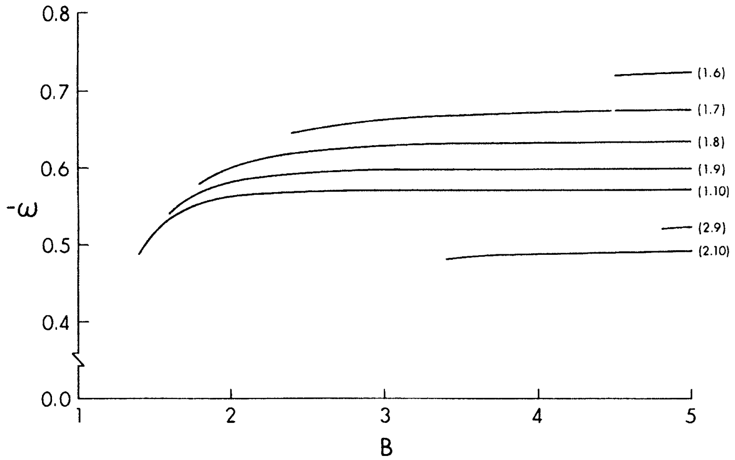

Fig. 4 (e). Same as Fig. 4 (a), but for $\delta=3$.

trapped modes. This descrepancy is attributed to the differences between models (1.1) and (1.3): the latter includes a realistic shoreline. In fact, SAINT-GUILY'S model (1.2) also has a shoreline and yields the dispersion relation with low wavenumber cut-off, in spite of the fact that his shape for the island topography is quite different from the present one. It is also interesting to note that in the case of a cylindrical island with vertical walls surrounded by water of constant depth (h_0), low-frequency Kelvin-type waves can be trapped if $n(n-1)<\mu^2$, which is a high wavenumber cut-off condition (LONGUET-HIGGINS, 1969).

In Fig. 3(a) and (b) are plotted the non-dimensional eigenfunctions Z, HU and HV representing sea level, radial and azimuthal volume transports for the first two modes corresponding to P and Q indicated in Fig. 2. Fig. 3 (a) and 3(b) show radial variation of the eigenfunctions for the first and second mode with the azimuthal wavenumber 6. Notice that the Z graph crosses the s-axis once in (a) but twice in (b). Thus, the number of crossings for the jth mode will be j. Both figures show the general characteristic of shelf waves: large longshore velocity in the vicinity of the shore and small sea level displacement. [Note that the scale of Z is enlarged by a factor of 20. For normalization A_{II} was taken to be unity.] It may be worthwhile to note that in the open ocaen ($s\geq B$) both velocity components (H being one in

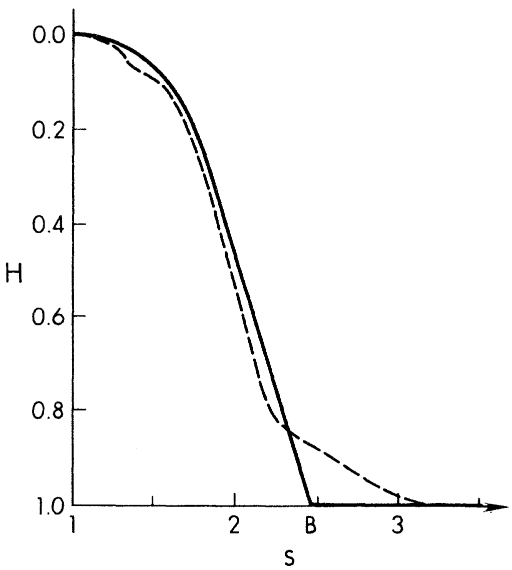

Fig. 5. Non-dimensional depth profile off Kahuku, island of Oahu (dotted line) and its approximation (solid line).

this region) rapidly tend to zero as (s) increases and that U and V have appproximately the same values with opposite signs. To see this more clearly, we use an approximate formula for a modified Bessel function with small argument

$$x : K_n(x) + \approx \frac{1}{2} P(n) (\frac{1}{2} x)^{-n}$$

(see, e.g., ABRAMOWITZ and STEGAN, 1972). Then from (2.8), (2.9) and (3.2) we obtain

$$Z \approx Cs^{-n},$$

$$(U,V) \approx (C, -C) \frac{n}{\mu(1-\omega)} s^{-n-1},$$

$$\text{for } s \geq B, \tag{3.9}$$

where

$$C = B^{-\ell-\delta/2+n} \ F(\ell+m, \ \ell-m, 1; \ 1-\frac{1}{B}).$$

The constant A_I is determined by the condition (3.6) with $A_{II} = 1$. Eq. (3.9) indicates that Z, U and H tend to zero as s^{-n} or s^{-n-1}, so that the shorter waves are more closely trapped by the island.

Fig. 4(a) shows the graphs of the nondimensional frequency as a function of B when $\delta = 0$ and $\mu = 0.01$. The symbol (j, n) represents the j'th mode with wavenumber n. In the figure there are three groups of curves corresponding to the first three modes. Each group consists of an infinite number of curves for each wavenumber n, but the solutions for n = 9 or larger are not shown. In each group, the lowest wavenumber is not 1 〔(1,1), (1.2), (2,1),...etc. are missing〕, reflecting the fact that the low wavenumber cut-off exists. It is seen that the frequency increases rather rapidly as B increases from 1 to 2, after which variation becomes gradual. This implies that the wave characteristic crucially depends on the shelf width when it is narrower than the island radius; but as the shelf becomes broader, the waves are less influenced by its width. As B becomes smaller, however,

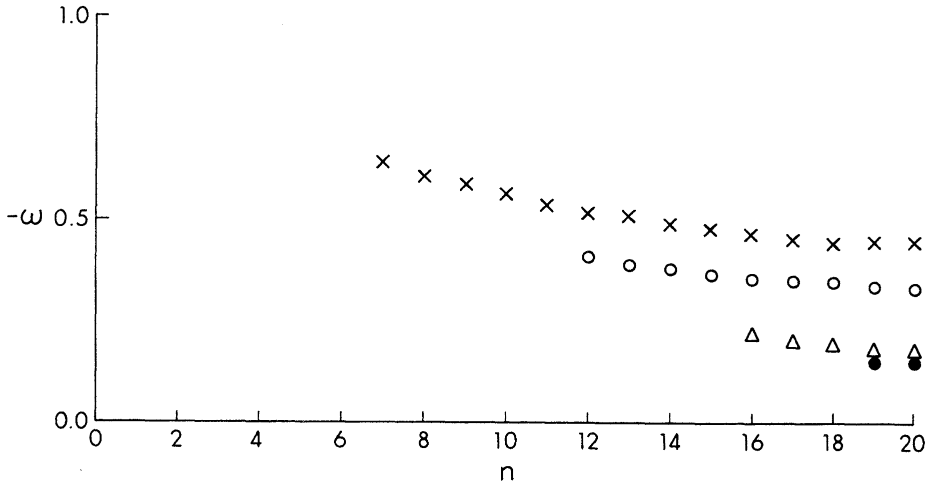

Fig. 6. Dispersion relation for the model island of Oahu.
〔× : 1st mode; ○ : 2nd mode; △ : 3rd mode; ● : 4th mode〕

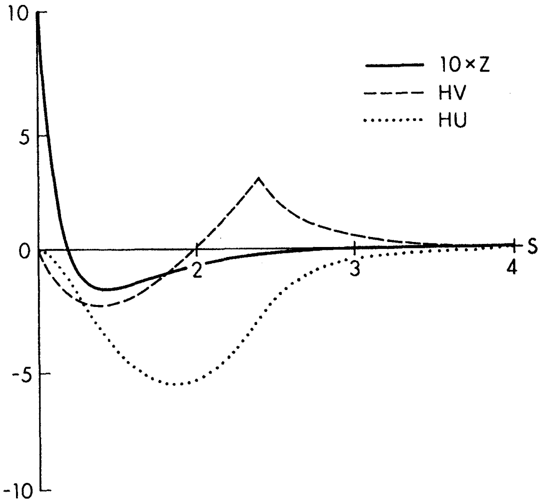

Fig. 7. Sea level and volume transports for the 1st mode wave with n=7 in Fig. 6.

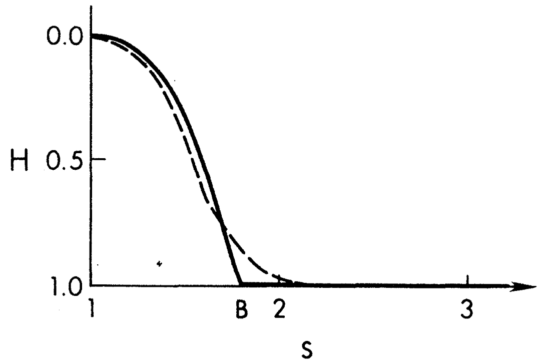

Fig. 8. Non-dimensional depth profile off Hilo, island of Hawaii (dotted line) and its approximation (solid line).

there are fewer solutions and only a high wavenumber spectrum is possible. That is, for a narrow and steep shelf, only short waves can be trappped.

The non-demensional frequencies depend also on δ which is a measure of the bottom slope. As seen from Fig. 4(a) through (e), the frequencies for greater δ have higher values. The low wavenumber cut-off also increases as δ increases, resulting in fewer curves in the graphs for greater δ. [In Fig. 4 (b) through (e), the solutions for n=11 or larger are not shown.] These figures are drawn for a fixed value of μ = 0.01; varying μ from 0.005 to 0.2 made no significant change in the results, which indicates that the effect of horizontal divergence can be neglected. For further confirmation, the solutions without assuming (3.3) in the shelf region were obtained for $\delta=1$ and 2. The solution (3.5) still holds except that now ℓ or m should be modified to:

$$\delta=1:\ \ell=\frac{1}{2}-\sqrt{n^2+\frac{2n}{\omega}+1+\mu^2(1-\omega^2)},$$

$$\delta=2:\ m=\sqrt{n^2+\frac{2n}{\omega}+1-\mu^2(1-\omega^2)}.$$

Calculations with these modified parameters with μ = 0.2 showed no appreciable difference either in eigenvalues or eigenfunctions. Thus, the rigid lid approximation (3.3) is justified.

Although the model discussed above may be too simple to be applied to real islands, one nevertheless might expect some trapped modes around roughly circular islands. To compare the present theory with the two-day oscillations found by MIYATA and GROVES (1968) around the island of Oahu, a typical depth profile offshore from Oahu is assumed by (2.11) for $\delta=4$ and B=2.4 (Fig. 5). The corresponding dispersion relationship (μ is taken to be 0.005) is shown in Fig. 6. As seen from the figure, the simplest possible trapped wave is the first mode with wavenumber 7. This wave has a non-dimensional frequency of 0.644, which corresponds to 0.47 cpd at the latitude of Oahu (21.5°). Thus, it is not unreasonable to identify this frequency with the observed two-day oscillation. LONGUET-HIGGINS (1971) offered the same kind of explanation using his model (1.1), but the reason for his selection of first mode with wavenumbers 4 and 5, rather than 1, 2 or 3, was not given. The eigenfunctions Z, HU, and HV corresponding to this particular wave are shown in Fig. 7.

As another example of application, the island of Hawaii is chosen and its typical offshore topography is plotted in Fig. 8. In this case, the computed dispersion relation (Fig. 9) differs slightly from that for Oahu, in accordance with the change of parameters ($\delta=4$, B=1.8, μ=0.01). The simplest possible trapped wave now is the first mode with wavenumber 8, at a non-dimensional frequency of 0.579. This is equivalent to 0.39 cpd, the latitude being taken as 19.7°. Thus, one should expect the trapped wave

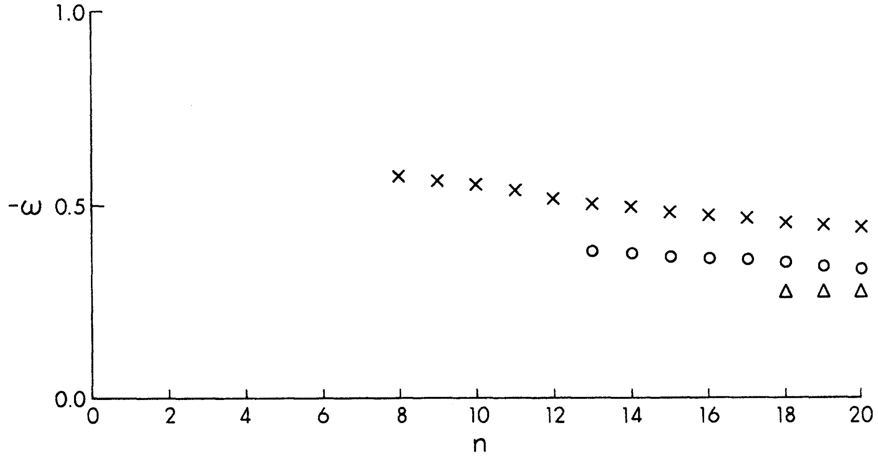

Fig. 9. Dispersion relation for the model island of Hawaii.
〔×: 1st mode; ○: 2nd mode; △: 3rd mode〕

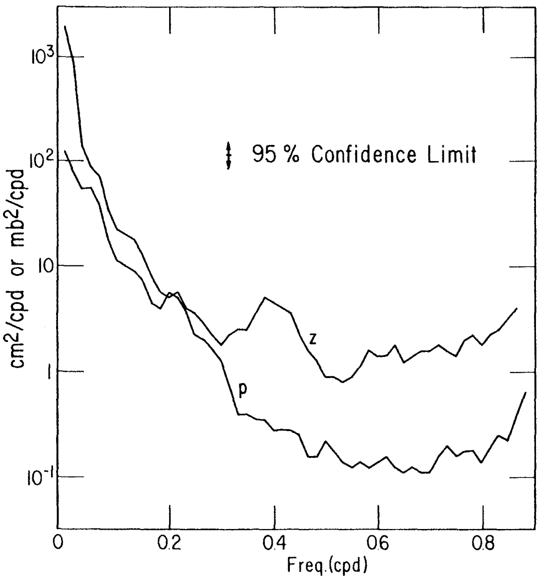

Fig. 10. Power spectra of Hilo sea level z and pressure p.
(MIYATA and GROVES, 1971).

with this frequency around Hawaii. The observed peak in the sea level spectrum at Hilo (Fig. 10) suggests the presence of such waves.

4. Conclusion

It has been shown that shelf waves trapped by an island with sloping beach and shoreline have a unique property: a low wavenumber cut-off for each radial mode. This property, together with the fact that the dispersion relation exists only for discrete wavenumbers, makes it easier to identify trapped waves from sea level spectra. The results are successfully applied to observations from the Hawaiian Islands.

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島のまわりの陸棚波

宮田元靖

要旨：島のまわりに補促される長周期波の理論モデルを論じた。海底地形に伴って種々の陸棚波モードが存在するが，各モードに対し，低波数側にカットオフが存在することが判明した。結果をハワイ諸島からの観測結果と比較した。