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# Amplification of long waves on a continental slope*

Motoyasu MIYATA**

**Abstract:**: A theoretical model is presented for investigating the effects of continental slope on tsunami run-up height. The model consists of three regions: I. Deep ocean of constant depth. II. Continental slope of varying topography. III. Beach and continental shelf of uniform slope. Using hydrostatic approximation, standing wave solutions over these regions are obtained and relative run-up heights are calculated. The results show that the topography of continental slope is an important factor to determine the run-up.

## I. Introduction

Since the first systematic work by KAPLAN (1955), tsunami run-up has been extensively studied especially in a laboratory (e.g. IWASAKI *et al.*, 1970; TOGASHI and NAKAMURA, 1978). Most of these experiments, however, were performed on a uniformly sloping beach connected to an open ocean of constant depth. This is not only because the topography is relatively simple but also because this is the only case for which an explicit theoretical solution is available (KELLER and KELLER, 1964; SHUTO, 1972).

It is certainly true that a uniformly sloping topography approximates almost any beach in nature and continental shelf within reasonable accuracy. However, beyond continental shelf break there usually exists a continental slope where the bottom slope increases. In the present paper, a theoretical analysis is made of a simple model which contains continental shelf topography, to provide information about the role of varying bottom on the tsunami run-up. Application of the theory to the real ocean will be considered in a separate paper using a numerical model.

Modelling and formulation will be explained in the following section. The solutions are given in Section III. Non-linear effect near shore will be discussed in Section IV.

## II. Formulation

The two dimensional linear unforced equations under the hydrostatic assumption in a homogeneous inviscid non-rotating ocean are given by:

$$\frac{\partial u'}{\partial t'} = -g\frac{\partial \zeta'}{\partial x'} \tag{2.1}$$

$$\frac{\partial \zeta}{\partial t'} + \frac{\partial (h'u')}{\partial x'} = 0 \tag{2.2}$$

Where the x axis is taken perpendicular to the straight coast line, u is the velocity in the off-shore direction, $\zeta'$ is the surface displacement, h' is the depth of the undisturbed ocean, and g is the acceleration of gravity. We introduce the non-dimensional variables as follows: $x' = \ell x$, $h' = Hh(x)$, $t' = \frac{\ell}{\sqrt{gH}}t$, where $\ell$ is the width of the continental shelf plus slope and H is the constant depth of the open ocean (see Fig. 1). Assuming perfect reflection at the coast, we seek the standing wave solutions:

$$(u', \zeta') = (\sqrt{gH}\,u,\ iH\zeta)e^{i\sigma t} \tag{2.3}$$

where u and $\zeta$ are functions of x only and $\sigma$ is the non-dimensional angular frequency. Then the equations (2.1) and (2.2) become:

$$\frac{d\zeta}{dx} + \sigma u = 0 \tag{2.4}$$

$$\sigma\zeta - \frac{d(hu)}{dx} = 0 \tag{2.5}$$

Combining these two equations yields:

$$\frac{d}{dx}\left(h\frac{d\zeta}{dx}\right) + \sigma^2\zeta = 0 \tag{2.6}$$

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** Department of Earth and Planetary Physics, University of Tokyo, 2-11-16 Yayoi, Bunkyo-ku, Tokyo, 113 Japan
Persent affiliation:
PICES, c/o Institute of Ocean Sciences, P. O. Box 6000, Sidney, B. C. Canada

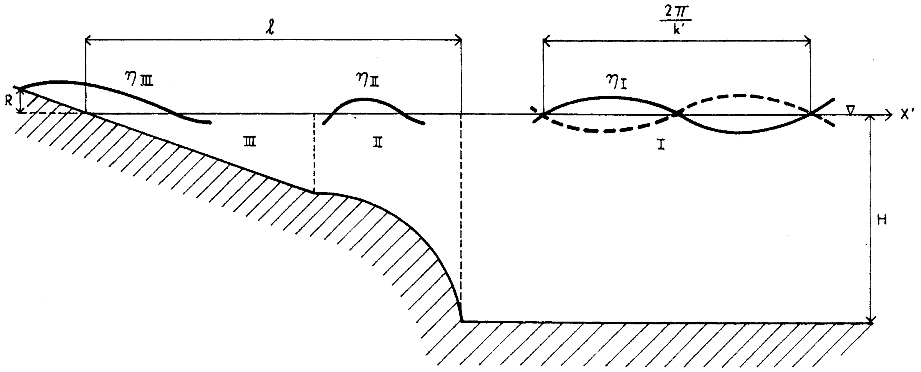


Fig. 1. Definition sketch.

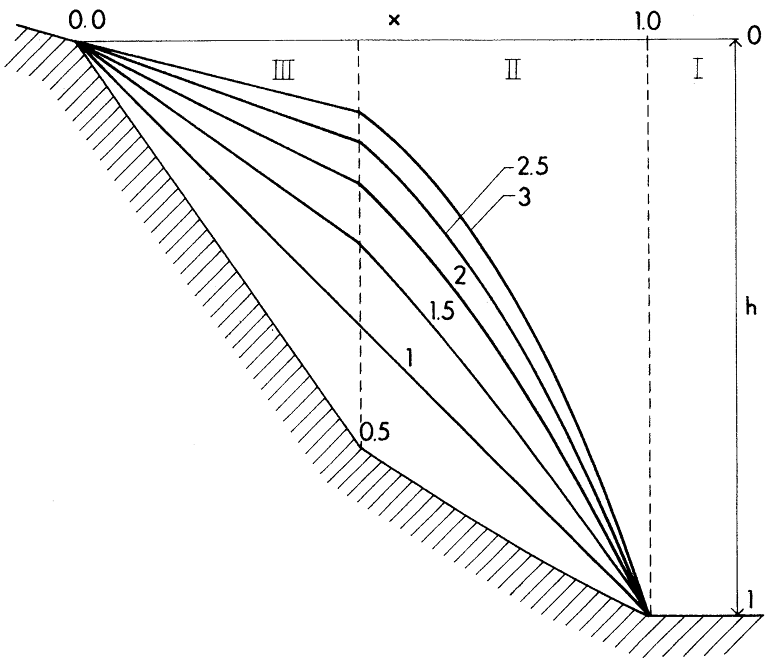


Fig. 2. Examples of bottom topography for $x_0$=0.5, $\delta$=0.5, 1, 1.5, 2, 2.5 and 3.

The following non-dimensional bottom topography is assumed.

$$h=\begin{cases}1 & 1\leqq x \quad \text{(Region I)}\\ x^{\delta} & x_0\leqq x<\ell \quad \text{(Region II)}\\ bx(b=x^{\delta-1}) & 0\leqq x<x_0 \quad \text{(Region III)}\end{cases} \tag{2.7}$$

where $\delta$ is an arbitrary constant and $x_0$ is the non-dimensional width of the continental shelf. (The width of the continental slope is $1-x_0$). Some example profiles are given in Fig. 2.

Now the problem into solve Eq. (2.6) in the three regions and match them at the boundaries.

## III. Solution

In Region I, Eq. (2.6) is simply

$$\frac{d^2\zeta}{dx^2}+\sigma^2\zeta=0 \tag{3.1}$$

The solution is

$$\zeta_{\mathrm{I}}=A_{\mathrm{I}}\cos(kx+\phi) \tag{3.2}$$

where k is the non-dimensional wave number and $\phi$ is the phase factor. Substituting (3.2) into

(3.1) gives the well-known relationship for shallow-water standing waves.

$$\sigma = k \tag{3.3}$$

In Region II, Eq. (2.6) can be rewritten as:

$$\frac{d^2\zeta}{dx^2}+\frac{\delta}{x}\frac{d\zeta}{dx}+\frac{k^2}{x^\delta}\zeta=0 \tag{3.4}$$

where $\sigma$ has been replaced by k.

For $\delta \fallingdotseq 2$, the solution of this equation is

$$\zeta_{II}=\begin{cases} x^{-1/2}(A_{II}x^p+B_{II}x^{-p}) & 1-4k^2\geqq 0 \\ x^{-1/2}\{A_{II}\cos(q\log x)+B_{II}\sin(q\log x)\} & 1-4k^2<0 \end{cases} \tag{3.5a}$$

where $p=1/2\sqrt{1-4k^2}$, $q=1/2\sqrt{4k^2-1}$

For $\delta \neq 2$, the solution can be expresses as follows:

$$\zeta_{II}=x^\delta\{A_{II}J_\nu(\beta x^\gamma)+B_{II}N_\nu(\beta x^\gamma)\} \tag{3.5b}$$

where $J_\nu$ and $N_\nu$ are Bessel functions of the first and second kind with order $\nu$. $\alpha$, $\beta$ and $\gamma$ are related to $\delta$ by:

$$\alpha=\frac{1-\delta}{2},\quad \beta^2=\left(\frac{2k}{2-\delta}\right)^2,\quad \nu^2=\left(\frac{1-\delta}{2-\delta}\right)^2$$

In Region I, the equation becomes:

$$\frac{d^2\zeta}{dx^2}+\frac{1}{x}\frac{d\zeta}{dx}+\frac{k^2}{bx}\zeta=0 \tag{3.6}$$

The solution in this case is simply:

$$\zeta_{III}=A_{III}J_0\left(2k\sqrt{\frac{x}{b}}\right) \tag{3.7}$$

Once the surface displacements are given by (3.2), (3.5) and (3.7), the velocity fields are known from Eq. (2.4).

At the boundaries x=1 and $x=x_0$, both displacement and velocity must be continuous, so that,

$$\zeta_I=\zeta_{II},\quad \frac{d\zeta_I}{dx}=\frac{d\zeta_{II}}{dx}\quad \text{at}\quad x=1 \tag{3.8}$$

$$\zeta_{II}=\zeta_{III},\quad \frac{d\zeta_{II}}{dx}=\frac{d\zeta_{III}}{dx}\quad \text{at}\quad x=x_0 \tag{3.9}$$

At the coast, the tsunami run-up height can be approximated by

$$R=\mathrm{Max}\ \{\zeta_{III}(0)\} \tag{3.10}$$

This approximation is consistent with the linear theory, and its validity will be further discussed in the next section.

From Eq. (3.8), (3.9) and (3.10) we can determine the phase factor $\phi$ and the tsunami run-up factor a defined by:

$$a=\frac{R}{A_I} \tag{3.11}$$

For $\delta=2$, and $4k^2\geqq 1$,

$$a=\left|\frac{k(x^p_0+sx_0^{-p})}{J_0(x_1)\sqrt{x_0\{k^2(1+s)^2+(m+ns)^2}}\right|$$

$$\phi=-k+\tan^{-1}\frac{k(1+s)}{m+ns}$$

where $m=\frac{1}{2}-p$, $n=-\frac{1}{2}-p$, and $x_1=2k\sqrt{\frac{x_0}{b}}$

$$s=\frac{B_{II}}{A_{II}}=-\frac{2mJ_0(x_1)+x_1J_1(x_1)}{2nJ_0(x_1)+x_1J_1(x_1)}$$

For $\delta=2$ and $4k^2<1$,

$$a=\left|\frac{k\{\cos(q\log x_0)+s\sin(q\log x_0)\}}{\sqrt{x_0\{k^2+(sq-1/2)^2\}}}\right|$$

$$\phi=-k+\tan^{-1}\frac{k}{sq-1/2}$$

where

$$s=\frac{B_{II}}{A_{II}}=\frac{\{1+2q\tan(q\log x_0)\}J_0(x_1)-x_1J_1(x_1)}{\{2q-\tan(q\log x_0)\}J_0(x_1)+x_1J_1(x_1)}$$

For $\delta \fallingdotseq 2$

$$a=\left|\frac{x^\alpha_0\sin(k+\phi)}{J_0(x_1)}\ \frac{J_\nu(\beta x_2)+sN_\nu(\beta x_2)}{J_0(\beta)+sN_\nu(\beta)},\right|$$

$$\phi=-k+\tan^{-1}\frac{k\{J_\nu(\beta)+sN_\nu(\beta)\}}{(\alpha+\nu\gamma)\{J_\nu(\beta)+sN_\nu(\beta)\}-\beta\gamma\{J_{\nu+1}(\beta)+sN_{\nu+1}(\beta)\}}$$

where $x_1=2k\sqrt{\frac{x_0}{b}}$, $x_2=x^\gamma_0$ and

$$s=\frac{B_{II}}{A_{II}}=\frac{2J_0(x_1)\{(\alpha+\nu\gamma)J_\nu(\beta x_2)-\gamma x_2J_{\nu+1}(\beta x_2)\}-x_1J_1(x_1)J_\nu(\beta x_2)}{2J_0(x_1)\{\gamma x_2N_{\nu+1}(\beta x_2)-(\alpha+\nu\gamma)N_\nu(\beta x_2)\}-x_1J_1(x_1)N_\nu(x_0)}$$

## IV. Discussions

The obtained results become particularly simple when $\delta$ is equal to unity. In this case since $\nu=0$, $N_\nu$ is no longer independent of $J_\nu$ so that we can put s=0: Then, by using b=1, $\sigma=0$, $\beta=2k$, and $\gamma=1/2$

$$a=\left|\frac{1}{J_0(2k)}\sin\tan^{-1}\frac{J_0(2k)}{2J_1(2k)}\right|$$

$$=\frac{1}{\sqrt{J_0^2(2k)+J_1^2(2k)}}$$

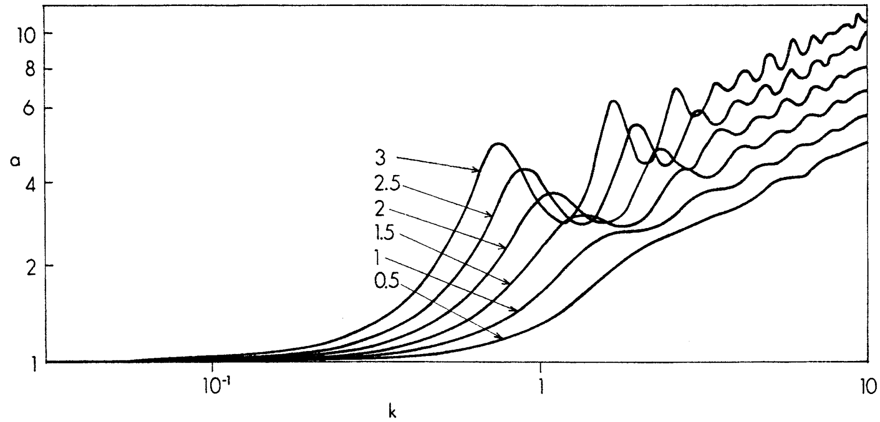


Fig. 3 Tsunami amplification factor for profiles in Fig. 2. The number indicates the value of $\delta$ in Eq. [2.7]. The curve for $\delta=1$ correspond to KELLER and KELLER (1964) result.

This result is in exact agreement with that of KELLER and KELLER (1964), as expected.

In Fig. 3, values of the run-up factor a are plotted as a function of k for $x_0=0.5$, $\delta=0.5$, 1, 1.5, 2, 2.5 and 3. It is seen that for small k, the factor a is near unity for any $\delta$, indicating that very long waves are not amplified at the coast. As k becomes larger, a tends to increase steadily but with minor oscillations. The oscillation is especially conspicuous for larger $\delta$. It is to be noted that although the overall slope between $x=0$ and $x=1$ is the same for every case (see Fig. 2), the run-up factors for larger $\delta$ are much higher than for lower $\delta$. This implies that the run-up height is determined more by the sea floor depth itself than its gradient: the shallower the continental shelf and slope, the greater the run-up. Approximating the bathymetry between the coast and the foot of the continental slope by a straight line may cause erroneous estimation of tsunami amplification.

The theory so far discussed is realistic for the region I and II where amplitudes of tsunami are usually small, but may not be valid near the coast where the non-linear effect becomes substantial. Fortunately, we have CARRIER and GREENSPANS' (1958) results of non-linear shallow water waves on a uniformly sloping beach and can examine the validity of the linear theory. It turns out that far from the coast, their solution approaches the linear solution given by (3.6). In the vicinity of the shoreline, the solutions significantly differs from the linear ones, but the maximum height to which the water rises on the shore is found to be simply: $R=A_{\mathrm{III}}$

So the non-linear theory would give approximately the same result as the linear one as far as the run-up factor is concerned. If the uniform slope extended to infinitely far, the two results would make no difference, as first pointed out by KELLER (1964). Therefore, our results are practically valid unless the shelf break $x_0$ is very near shore. It is not surprising that numerical experiments including non-linear terms show fair agreement with the linear theory as shown by GOTO (1970) for the case of $\delta=1$.

## V. Conclusions

It is found by using a simple theoretical model that the varying bathymetry of the continental slope may sometimes produce much higher tsunami run-up than the continental shelf uniformly sloping down to the open ocean. The theory remains to be tested by laboratory experiments.

## Acknowldgement

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# 大陸斜面上における長波の増幅

宮 田 元 靖

**要旨**：大陸斜面が津波の遡上高にどのような影響を与えるかを，簡単なモデルを用いて理論的に考察した。モデルは3つの領域から成る。即ち，一定水深の深海領域，可変水深の大陸斜面領域，及び一定勾配の大陸棚と海岸領域である。静水圧近似のもとで，これらの領域における定在波の解を求め，振幅の増幅率を計算し比較した。津波の遡上高に対して，大陸斜面が重要な役割を果たすことが示された。